Mathematics of Computing
Final Exam (Max Marks 50, Time 3h)

Indian Statistical Institute, Bangalore

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NB: (a) E_{TM} is the set of Turing machines $< M >$ whose language is the empty language. (b) EQ_{TM} is the set consisting of pairs $< M_1, M_2 >$ of Turing machines which recognize the same language. (c) A_{TM} is the set consisting of all $< M, w >$ pairs where the Turing machine M accepts the string w. (d) A *Vertex Cover* of size k of a graph G is a k size subset of the vertex set of G such that every edge has at least one end in it.

1. (1x10=10) Assume $A \leq_P B$ i.e., A reduces to B in polynomial time by a deterministic Turing machine. Mention True or False for each:
   (a) If A is Turing decidable then so is B.
   (b) If A is Turing recognizable then so B.
   (c) If A is not Turing decidable then B is not.
   (d) $B \leq_P A$
   (e) If A is polynomial time Turing decidable then so is B.
   (f) If A is NP-complete then so is B.
   (g) If B has an exponential time algorithm then so does A.
   (h) If B is both Turing recognizable and co-Turing recognizable, then A is decidable.
   (i) EQ_{TM} is neither Turing recognizable nor co-Turing recognizable.
   (j) E_{TM} is Turing recognizable but not co-Turing recognizable

2. $(3 + 3 + 2 + 2 = 10)$
   (a) Draw a DFA that accepts all non-negative integer multiples of 6 written in the usual decimal base, with possibly leading zeros. You may use the fact that a number is divisible by 6 iff it is even and divisible by 3.
   (b) Convert the following regular expression to an NFA and then, convert the NFA to a DFA using subset construction: 0(0|1)*00.
   (c) State the *pumping lemma* for regular languages.
   (d) Show that the language $L = \{0^n 1^m | n > m\}$ is not regular.

3. $(2 + 2 + 2 + 4 = 10)$

(a) Consider the language $L = \{$ w is a palindrome on $\{0, 1\}$ of odd length$\}$. Give an NPDA for L (graphically or transition table).

(b) For the language in part (a) give a CFG.

(c) State the *pumping lemma* for context free languages.

(d) Prove that the language $L = \{ww|w$ is a binary string$\}$ is not context free.

4. $(4 + 3 + 3 = 10)$ You may assume, as proved in class, that A_{TM} is Turing recognizable and not Turing decidable.

   (a) We learnt that $A_{TM} \leq_M Th(N, +, \times)$. Given that proof systems are sound and that proofs are verifiable, show that $Th(N, +, \times)$ is Turing recognizable.

   (b) From the result in part (a) show that there exist some statements in $Th(N, +, \times)$ that are true, however not provable.

   (c) We know that if $A \leq_m B$, then if A is not Turing recognizable then B is not Turing recognizable. Use this to show that EQ_{TM} is not Turing recognizable.

5. $(1 + 2 + 3 + 4 = 10)$

   (a) We say that if a problem has a certificate of membership that is checkable in deterministic polynomial time, then the problem is in NP. How is this equivalent to saying that the problem is decided by non-deterministic polynomial time Turing machine?

   (b) Define the terms *NP-Hard* and *NP-Complete*.

   (c) The language CLIQUE contains all pairs $< G, k >$ where the graph G has k vertices which form a complete subgraph. Let G have n vertices. Prove or disprove this claim: Since the number of ways of choosing k vertices from n is no more than $O(n^k)$, we can check all of them, and it takes no more than $O(n^k)$ time, which is polynomial in n; thus CLIQUE is in P.

   (d) Let $S = \{a_1, a_2, ..., a_n\}$ be a set of n elements. Let $X = \{s_1, s_2, ...s_m\}$ be a collection of m subsets of S. As an example s_1 may be $\{a_1, a_3, a_5, a_6\}$. For a given S and X we are interested in a subset H of S so that H has a non-empty intersection with each s_i in X. Such a set H is called a *Hitting Set* for the given S, X pair. Show that the problem of determining if a given S, X pair has a hitting set of a given size k is NP-Complete. For the reduction use the fact that Vertex Cover is NP-Complete.